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Drag due to the motion of a Newtonian fluid through a sparse random array of small fixed rigid objects

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The averaged equations of slow flow in random arrays of fixed spheres are developed as a hierarchy of integro-differential equations, and an iteration procedure is described for obtaining the mean drag in the case of small volume concentration c. The leading approximation is that given by Brinkman's model of flow past a single fixed sphere, in which the effects of all other spheres are treated as a Darcy resistance. The higher approximations take account of the modification to the mean flow, particularly in the near field, due to the localized nature of the actual resistance. Thus the second approximation finds the change due to a second sphere, and averages over all its possible positions. The result for the mean drag confirms Childress' terms in $c \log c$ and c (apart from an arithmetical correction to the latter), but indicates that for practical values of c numerical evaluation of integrals is needed, rather than expansion in powers of c and $\log c$. The last section of the paper develops the corresponding results for flow through random arrays of fixed parallel circular cylinders.

1. Introduction

This paper has as its context two different problems of viscous flow through sparse random arrays of small rigid objects, subject to external forces.

(*a*) The sedimentation problem, where the force on each object is prescribed, hence the velocities of the objects are random, and the mean sedimentation velocity is to be found (case II in Saffman 1973).

(*b*) The porous-medium problem, where the objects are at rest, or have prescribed velocities, so that the forces acting on them must be random, and the mean drag force is to be found (case III in Saffman 1973).

The method set out by Batchelor (1972), recognizing that it is not sensible to try to sum all the individual flow fields from isolated spheres in a uniform flow, shows that the need to do so is removed by proper use of the bulk properties of the flow. This has overcome the trouble with divergent integrals in problem (*a*). But in problem (*b*) the same idea must be taken further, with the introduction of a modified differential equation that can account not just for the bulk properties but also for flow variations having length scales much larger than the sphere radius.

The distinctive feature of the latter problem, namely that the resistance due to the whole cloud of particles enters crucially into the flow past an individual particle, was incorporated into a mathematical model by Brinkman (1947), for the case of small spherical objects. To describe the mean flow past a particular sphere he set up an effective medium, defined by the simplest differential equation that would reduce to Darcy's law for a uniform velocity and to the equation of slow viscous flow for small-scale velocity variations. His reasoning is heuristic, but somewhat surprisingly, the resulting first correction to the simple Stokes drag formula, for sparse distributions of spheres, is exactly correct. A reason for this is suggested at the end of § 3 of this paper. It should be emphasized that Brinkman's model is to be viewed as providing a first approximation for the mean flow past a sphere in the case of sparse distributions (and perhaps more generally). It is basic to the present paper, which seeks to give a scheme for obtaining higher approximations.

Tam (1969), in addition to generalizing Brinkman's result to the case of a distribution of sphere sizes, also set out to derive Brinkman's equation, in the point-particle approximation for the spheres. But the estimate he gives for the error is incorrect, and attention should also be paid to the question of convergence of the various sums and integrals used as the number of particles tends to infinity (which is the situation presupposed by the final equations for drag). In some cases the convergence will fail.

The article by Lundgren (1972) studies more fully the flow past a sphere, subject to Brinkman's equation. But in the application of this to the porous-medium problem, it ignores more important effects due to the difference between distributed and localized resistance.

Childress (1972) first develops an elaborate scheme for ordering all the interactions between spheres, thus calculating the drag correct to relative order c. Then in § 7 he gives an alternative method using the point-particle approximation, based on an infinite hierarchy of differential–functional equations for the mean flows in the cases where $1, 2, 3, \ldots$, particles have prescribed positions. The use of differential equations from the start avoids the convergence difficulty, and by going to the second equation of the set before making the closure approximation he is able to carry the exact expansion for the drag a stage further than Brinkman and Tam. The results obtained by his two methods agree as far as the term in $c \log c$; more could not be expected because of the point-particle approximation.

The second method, much the simpler of the two, is limited by Childress to the point-particle model since if random regions are to be omitted in averaging the velocity field, then the operation of differentiating with respect to the point of observation does not commute with the averaging. But this is not a necessary limitation: in the present paper the difficulty is avoided by the use of an extended velocity field, obeying the viscous flow equation even in the interior of the obstacles. We then rewrite Childress' hierarchy by introducing the mean surface stress distribution over an obstacle in a prescribed position. An iteration method of solution is suggested, and illustrated by calculating the drag to relative order $c^{\frac{3}{2}}$. This claim to accuracy is based on reasonable error estimates which the formulation makes possible.

Two-dimensional flows (both transverse and longitudinal), through random arrays of infinite parallel cylindrical rods, are treated by the same method. In these problems, however, the effect of the fixed obstacles is more striking. For in contrast to the three-dimensional case, there would be no solution for flow past a cylinder, given uniform flow at infinity, without some modification of the governing equation. The relative orders of magnitude of effects are also considerably changed, and (as might be expected) the drag expansions contain sub-series in negative powers of $\log c$ before powers of c are encountered.

The layout of the article is as follows. The averaging of the viscous flow equations is explained in § 2. Section 3 contains further preliminary discussion of the whole question, in terms of the elementary mean flows, or Green's functions. In § 4 the most convenient approximations to these are derived as solutions of Brinkman's equation, as well as drag formulae for flows satisfying that equation. Section 5 develops the hierarchy of equations for the mean flow through an infinite array of spheres and for the mean drag on the spheres. In § 6 the mean drag is calculated as far as the term in $c^{\frac{3}{2}}$, and § 7 contains the corresponding calculation for arrays of infinite cylinders.

2. Notation and formulation of the basic mean flow equations

Spherical objects

In the simplest situation, the fixed rigid spheres, all of radius a, are distributed in a statistically homogeneous manner, with mean number density n per unit volume, in an unbounded Newtonian fluid of viscosity μ. Then $c = \frac{4}{3}\pi a^3 n$ is the mean volume fraction occupied by the spheres.

Mean values are taken with respect to the ensemble of configurations of this infinite array. Equations will be written for the mean values of flow quantities at the point $\mathbf{r}$, conditional on the presence of a finite number of test spheres centred at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Probability densities, absolute and conditional, are denoted by (for example) $P(\mathbf{r})$ for the distribution of a single sphere centre and $P(\mathbf{r}|\mathbf{r}_1)$ for the conditional distribution given another sphere centred at $\mathbf{r}_1$. All the calculations are made using the simplest forms for these densities: the value n for $P(\mathbf{r})$, and for $P(\mathbf{r}|\mathbf{r}_1)$ when $|\mathbf{r}-\mathbf{r}_1| > 2a$; zero for $P(\mathbf{r}|\mathbf{r}_1)$ when $|\mathbf{r}-\mathbf{r}_1| < 2a$.

Before averaging, we need to extend the domain of definition of the flow quantities throughout the whole space (cf. Lundgren 1972, § 2); since the spheres are all at rest, a natural way of doing this is to make the velocity and pressure gradient zero in the interiors of the spheres, thus satisfying the same equation of slow viscous flow as in the exterior region. The extended velocity field is then continuous everywhere, and without sources or sinks, so that the averaged velocity also satisfies the incompressibility condition. In assigning values to the pressure itself, the only criterion is that statistical homogeneity be preserved, as it will be if we set the pressure in any sphere equal to the mean value at the position of its centre (mean with respect to all configurations for which that position lies in the fluid). The interior pressure is thus linearly related to the position of the sphere centre [equation (2.4)].

When the equation of slow viscous flow is averaged, the discontinuities in shear stress and pressure across the sphere surfaces contribute a resistance term, equal to the expected surface force per unit volume. Thus, however the averaging is conditioned, we can write for the equation of mean flow

$$-\nabla\langle p\rangle+\mu\nabla^2\langle \mathbf{u}\rangle = -\mathbf{R}(\mathbf{r}). \tag{2.1}$$

The distributed resistance $\mathbf{R}$ is given by an integral over the locus of the centre positions of spheres passing through $\mathbf{r}$. If $\mathbf{a}$ denotes a vector of length a, $d\mathbf{a}$ the element of area on the surface and we introduce the vector quantity $\mathbf{T}(\mathbf{a};\mathbf{r}-\mathbf{a},\ldots)$ for the mean surface stress at $\mathbf{r}$ subject to the additional condition of a sphere centred at $\mathbf{r}-\mathbf{a}$, then

$$\mathbf{R}(\mathbf{r}) = -\int \mathbf{T}(\mathbf{a};\mathbf{r}-\mathbf{a},\ldots)\,P(\mathbf{r}-\mathbf{a}|\ldots)\,d\mathbf{a}. \tag{2.2}$$

It is to be noticed that (2.1) holds even within the test spheres, and that $\mathbf{R}$ is zero there, as well as $\langle\mathbf{u}\rangle$ and $\nabla\langle p\rangle$.

Consider now the means in the absence of test spheres (except that when it comes to surface stress and total drag one test sphere is necessarily involved; this is indicated by the suffix 1). $\mathbf{U}$ will denote the mean velocity and $\mathbf{F}_1$ the quantity to be determined, the mean drag force on a sphere, which is related to the mean surface stress by

$$\mathbf{F}_1 = \int \mathbf{T}_1(\mathbf{a};\mathbf{r})\,d\mathbf{a}. \tag{2.3}$$

But given the extended definition of pressure, $\mathbf{T}_1(\mathbf{a};\mathbf{r})$ is independent of $\mathbf{r}$, and (2.2) shows that the (unconditioned) mean resistance force per unit volume is minus the number density times $\mathbf{F}_1$. Then (2.1) gives, with appeal to the linearity of the problem,

$$-\nabla\langle p\rangle = n\mathbf{F}_1 = \mu\alpha^2\mathbf{U}. \tag{2.4}$$

Here the notation $\mu\alpha^2$ has been introduced for the Darcy's law coefficient in (2.4), because of its greater convenience in the formulae for the Green's function (§4). The coefficient will eventually be determined by self-consistency. In terms of the mean effective velocity $\mathbf{U}_e$ felt by a sphere, $\mathbf{F}_1$ is $6\pi\mu a\mathbf{U}_e$, and

$$\alpha^2 = 6\pi a n U_e/U = (9c/2a^2)\,U_e/U. \tag{2.5}$$

The leading approximation to this, for small c, is

$$\alpha_0^2 = 9c/2a^2. \tag{2.6}$$

α^{-1} has the dimensions of length and is called the shielding radius.

In the presence of one or more test spheres, if the resistance force at each point is again set, as a first approximation, equal to $-\mu\alpha^2$ times the mean velocity there, then (2.1) leads to Brinkman's equation, which is studied in greater detail in §4. Actually Brinkman used a modified viscosity to allow for the effects of the finite size of the spheres, as also did Lundgren; we do not follow them in this, because the difference is of relative order c, and is more conveniently allowed for, together with other effects at least as important, by successive approximations based on the Green's function described in §4.

A final word about notation: second-order tensors are written throughout as dyadics, and indices denote powers as with matrices. (The unit dyadic $\mathbf{I}$ corresponds to δ_{ij}, and the direct product $\nabla\nabla$ to $\partial^2/\partial r_i \partial r_j$.) The tensor

$$\mathscr{I}(\mathbf{r}) = (\nabla\nabla - \mathbf{I}\nabla^2)\, 2\alpha^{-2} r^{-1}(1 - e^{-\alpha r})$$

gives the elementary solution for shielded Stokes flow (§ 4), in the sense that a point force $\mathbf{F}$ at the origin produces a velocity $(8\pi\mu)^{-1}\mathbf{F}.\mathscr{I}(\mathbf{r})$ at the point $\mathbf{r}$. Similarly $\mathscr{I}_0(\mathbf{r}) = -(\nabla\nabla - \mathbf{I}\nabla^2)\, r$ gives the ordinary Stokeslet.

The work on arrays of parallel rods requires only minor changes in the notation; these are noted in the relevant section.

3. Elementary flow fields and effective media

The modification of the flow equation by the resistance term $\mathbf{R}$ [equation (2.1)], and the difference between the sedimentation and porous-medium problems, can be usefully discussed in terms of the elementary velocity field, or Green's function: that is, the mean flow produced by a point force. This terminology is also related to the concept of an effective medium.

In the sedimentation problem the introduction of a point force into the flow makes no difference to the drag on any particle, so that the elementary flow is given to a good approximation by the ordinary Stokes flow with a modified viscosity. Thus the effective medium for the motion of one particle is still just a viscous fluid, though for closer approximations some non-uniformity of properties has to come in. But when the particles are all fixed, the introduction of a point force alters the drag on each in such a way that the fluid further out is shielded from the influence of the force. The effects of such strong interactions cannot be treated as confined to a small number of particles at a time, in a successive approximation process. They are most simply allowed for by Brinkman's uniform effective medium, having, as well as viscosity, a Darcy resistance coefficient to be determined by self-consistency [equation (2.5)]: the resulting elementary flow is the 'shielded Stokeslet' $\mathbf{S}.\mathscr{I}(\mathbf{r})$ as in equation (4.7) below. This is a good first approximation; that it is not exact can be seen by taking the surface stress on two fixed spheres when all the others are replaced by the uniform effective medium, and then using (2.2) to find the mean resistance field for the flows with one sphere in a prescribed position. The result cannot be proportional to the Brinkman form of the velocity field, on account of the second- and higher-order reflexions between the spheres. The second approximation should correct for these reflexions, and higher approximations take account of reflexions between increasing numbers of spheres.

For the purpose of making these corrections, § 5 generalizes the concept of a Green's function to that of the m-particle difference flow: the overall flow past m particles in any medium can be written as the sum of the main flow and a collection of such difference flows, one for each combination of some or all of the particles. Associated with these are the difference fields of resistance force, each related by (2.2) to a difference surface stress on an extra fixed particle introduced into the flow. Use of this relation means that the solution for k particles in one medium can define a problem with $k-1$ particles in a new medium, and so spheres is independent of the radius, but the total shearing force tends to zero at large radii, the variation being balanced by the distributed resistance.

Analogues of Faxen's formula

Some generalization of the drag result (4.6) is possible when the flow can be regarded as made up of a known, externally produced field, together with that generated directly by the stress distribution on the surface $r = a$, without any further reflexions:

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}_{\text{ext}}(\mathbf{r}) + \mathbf{u}_T(\mathbf{r}). \tag{4.9}$$

$\mathbf{u}_{\text{ext}}$, the externally produced field, has a regular continuation, satisfying (4.1), throughout the sphere $r \leqslant a$. $\mathbf{u}_T$, in $r > a$, is the flow generated by the (unknown) stress distribution $\mathbf{T}(\mathbf{a})$ [notation as in (2.2)]; it can be expressed in terms of the Green's function defined in (4.7), exactly as for ordinary slow viscous flows:

$$\mathbf{u}_T(\mathbf{r}) = -\frac{1}{8\pi\mu}\int \mathbf{T}(\mathbf{a}) . \mathscr{I}(\mathbf{r}-\mathbf{a})\, da. \tag{4.10}$$

We thus have a definition of $\mathbf{u}_T$ which is continuous throughout the whole space, is regular and satisfies (4.1) everywhere except on the surface $r = a$ (if the appropriate pressure fields are introduced). And so, within the sphere,

$$\mathbf{u} = \mathbf{u}_{\text{ext}} + \mathbf{u}_T$$

is a regular function satisfying (4.1); it vanishes on the boundary by continuity, and hence throughout the sphere.

The stress $\mathbf{T}(\mathbf{a})$ can then be determined by the requirement that $\mathbf{u}(\mathbf{r})$, together with all its derivatives, should be zero at $\mathbf{r} = 0$; but all that is needed to obtain the total drag is the vanishing of $\mathbf{u}(0)$ and $\nabla^2\mathbf{u}(0)$. Thus

$$\int \mathbf{T}(\mathbf{a}) . \mathscr{I}(\mathbf{a})\, da = 8\pi\mu\mathbf{u}_{\text{ext}}(0),$$
$$\int \mathbf{T}(\mathbf{a}) . \nabla^2\mathscr{I}(\mathbf{a})\, da = 8\pi\mu\nabla^2\mathbf{u}_{\text{ext}}(0),$$

from which, if $\mathbf{F}$ is the total drag, $\mathbf{F}_p$ that due to pressure forces and $\mathbf{F}_s$ that due to shearing forces, we find

$$\left.\begin{aligned} \mathbf{F}_s &= 4\pi\mu a\{1+\alpha a+(e^{\alpha a}-1-\alpha a)\,\alpha^{-2}\nabla^2\}\,\mathbf{u}_{\text{ext}}(0),\\ \mathbf{F}_p &= \tfrac{1}{2}\mathbf{F}_s + 2\pi\mu\alpha^2 a^3(1-\alpha^{-2}\nabla^2)\,\mathbf{u}_{\text{ext}}(0),\\ \text{and}\quad \mathbf{F} &= \mathbf{F}_s+\mathbf{F}_p\\ &= 6\pi\mu a\{1+\alpha a+\tfrac{1}{3}\alpha^2a^2+(e^{\alpha a}-1-\alpha a-\tfrac{1}{3}\alpha^2a^2)\,\alpha^{-2}\nabla^2\}\,\mathbf{u}_{\text{ext}}(0), \end{aligned}\right\} \tag{4.11}$$

the last of which could also be obtained from (4.5) by the principle of reciprocity. It may be noted that these lead to Faxen's formula for $\mathbf{F}$, as $\alpha \to 0$.

Further explicit results are possible in the case of the spherical symmetry introduced at the beginning of this section. For then the stress distribution is completely determined by the (parallel) vectors $\mathbf{F}_p$ and $\mathbf{F}_s$:

$$\mathbf{T}(\mathbf{a}) = (3/8\pi a^2)\,\{\mathbf{F}_s + \hat{\mathbf{a}}\hat{\mathbf{a}} . (2\mathbf{F}_p - \mathbf{F}_s)\};$$

and the velocity $\mathbf{u}_T$ can be evaluated: in $r > a$,

$$\begin{aligned}\mathbf{u}_T(\mathbf{r}) = -\tfrac{3}{4}a\left[(1+\alpha a+\tfrac{1}{3}\alpha^2a^2)\left(1-\frac{1}{\alpha^2}\nabla^2\right)\mathbf{u}_{\mathrm{ext}}(0)+e^{\alpha a}\frac{1}{\alpha^2}\nabla^2\mathbf{u}_{\mathrm{ext}}(0)\right]\\ \times(\nabla\nabla-\mathbf{I}\nabla^2)\frac{2}{\alpha^2 r}+\tfrac{3}{4}a\left[e^{\alpha a}\left(1-\frac{1}{\alpha^2}\nabla^2\right)\mathbf{u}_{\mathrm{ext}}(0)+\frac{e^{2\alpha a}-1}{2\alpha^3 a}\nabla^2\mathbf{u}_{\mathrm{ext}}(0)\right]\\ \times(\nabla\nabla-\mathbf{I}\nabla^2)\frac{2}{\alpha^2 r}e^{-\alpha r}.\end{aligned} \qquad (4.12)$$

Flows with cylindrical symmetry

For transverse flows (in parallel planes perpendicular to the axis of symmetry) there is a set of formulae corresponding closely to those given already in this section. For longitudinal flows (parallel to the axis of symmetry) the situation is rather simpler, because the velocity field is now effectively scalar, and also the pressure is constant in planes perpendicular to the axis. The main formulae in the two cases are set out in table 1 (r now denotes the two-dimensional radius and $\mathbf{F}$, $\mathbf{F}_s$ and $\mathbf{F}_p$ are forces per unit length).

Transverse flow	Longitudinal flow
Disturbance flow due to a rigid cylinder in uniform flow at infinity [cf. (4.2)–(4.6)]	
$\mathbf{u} = \mathbf{V}.(\nabla\nabla-\mathbf{I}\nabla^2) \times\{-A\alpha^{-2}\log\alpha r-B\alpha^{-2}K_0(\alpha r)\}$	$u = VK_0(\alpha r)$
$p = \mu\mathbf{V}.\nabla(A\log\alpha r)$	$p =$ constant
$\mathbf{V} = -\mathbf{U}/K_0(\alpha a)$	$V = -U/K_0(\alpha a)$
$A = 2\alpha aK_1(\alpha a)+\alpha^2a^2K_0(\alpha a)$,	
$B = 2$	
$\mathbf{F} = 4\pi\mu\{\alpha aK_1(\alpha a)/K_0(\alpha a)+\frac{1}{2}\alpha^2a^2\}\,\mathbf{U}$	$F = 2\pi\mu U\alpha aK_1(\alpha a)/K_0(\alpha a)$
Green's function [cf. (4.7)]	
$\mathbf{u} = \mathbf{V}.\mathscr{I}_t(\mathbf{r})$	$u = VK_0(\alpha r)$
$\mathscr{I}_t(\mathbf{r}) = (\nabla\nabla-\mathbf{I}\nabla^2) \times\{-2\alpha^{-2}\log\alpha r-2\alpha^{-2}K_0(\alpha r)\}$	
$\mathbf{F} = -4\pi\mu\mathbf{V}$	$F = -2\pi\mu V$
Faxen's formulae [cf. (4.11)]	
$\mathbf{F}_s = \dfrac{2\pi\mu}{K_0(\alpha a)}\left\{\alpha aK_1(\alpha a)\left(1-\dfrac{1}{\alpha^2}\nabla^2\right)\mathbf{u}_{\mathrm{ext}}(0)+\dfrac{1}{\alpha^2}\nabla^2\mathbf{u}_{\mathrm{ext}}(0)\right\}$	$F_s = \dfrac{2\pi\mu}{K_0(\alpha a)}[u_{\mathrm{ext}}(0)+\{\alpha aK_1(\alpha a)-1\}\,U]$
$\mathbf{F}_p = \mathbf{F}_s+2\pi\mu\alpha^2a^2(1-\alpha^{-2}\nabla^2)\,\mathbf{u}_{\mathrm{ext}}(0)$	$F_p = 0$ U is the flow due to the uniform axial pressure gradient, if any

TABLE 1

5. Scheme for the treatment of any number of test spheres

In the general case, there will be test spheres centred at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Then in the region exterior to all of them, the conditioned averages of the flow quantities (averaged over the positions of all the other spheres) satisfy (2.1), which we write in the form

$$-\nabla\langle p\rangle + \mu\nabla^2\langle \mathbf{u}\rangle - \mu\alpha^2\langle \mathbf{u}\rangle = -\mathbf{R}(\mathbf{r}) - \mu\alpha^2\langle \mathbf{u}\rangle. \tag{5.1}$$

The boundary conditions are that $\langle \mathbf{u}\rangle$ vanishes on $|\mathbf{r}-\mathbf{r}_k| = a$, $k = 1, 2, \ldots, m$, and tends to $\mathbf{U}$, the overall mean velocity, as r tends to infinity. But also, because of the extended definition of $\mathbf{u}$ throughout the whole space, its mean value vanishes in the interior of each test sphere, and hence (5.1) holds there too, with $\mathbf{R}$ and $\nabla\langle p\rangle$ equal to zero. The mean velocity can be written in the integral form, valid everywhere [cf. (4.9) and (4.10)],

$$\langle \mathbf{u}\rangle = \mathbf{U} - \frac{1}{8\pi\mu}\sum_{k=1}^{m}\int \mathbf{T}(\mathbf{a};\mathbf{r}_k).\mathscr{I}(\mathbf{r}-\mathbf{r}_k-\mathbf{a})\,da + \frac{1}{8\pi\mu}\int\{\mathbf{R}(\mathbf{r}') + \mu\alpha^2\langle \mathbf{u}(\mathbf{r}')\rangle\} \times \mathscr{I}(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}', \tag{5.2}$$

where $\mathbf{T}(\mathbf{a};\mathbf{r}_k)$ is the surface stress at $\mathbf{r}_k + \mathbf{a}$.

The quantities $\langle \mathbf{u}\rangle$, $\langle p\rangle$ and $\mathbf{R}$ are now to be expressed as sums of terms referring to successively larger numbers of test spheres, up to m. All terms (apart from those referring to the case of no test spheres) are to tend to zero at infinity.

In the absence of test spheres, there is no dependence on $\mathbf{r}$, and both sides of (5.1) vanish [see (2.4)]:

$$\langle \mathbf{u}\rangle = \mathbf{U}, \quad \mathbf{R}(\mathbf{r}) = -\mu\alpha^2\mathbf{U}, \quad \nabla\langle p\rangle = -\mu\alpha^2\mathbf{U}.$$

With one test sphere, centred at $\mathbf{r}_1$,

$$\langle \mathbf{u}\rangle = \mathbf{U} + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_1), \quad \mathbf{R} = -\mu\alpha^2\mathbf{U} + \mathbf{R}_1(\mathbf{r}-\mathbf{r}_1), \quad \nabla\langle p\rangle = -\mu\alpha^2\mathbf{U} + \nabla p_1(\mathbf{r}-\mathbf{r}_1).$$

Then, in $|\mathbf{r}-\mathbf{r}_1| \leqslant a$,

$$\mathbf{U} + \mathbf{U}_1 = 0, \quad -\mu\alpha^2\mathbf{U} + \mathbf{R}_1 = 0, \quad -\mu\alpha^2\mathbf{U} + \nabla p_1 = 0. \tag{5.3}$$

The mean surface stress is $\mathbf{T}_1(\mathbf{a})$ at $\mathbf{r}_1 + \mathbf{a}$, on the test sphere.

With two test spheres, centred at $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$\langle \mathbf{u}\rangle = \mathbf{U} + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_1) + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_2) + \mathbf{U}_2(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2),$$

with similar expressions for $\mathbf{R}$ and $\nabla\langle p\rangle$. In $|\mathbf{r}-\mathbf{r}_1| \leqslant a$,

$$\mathbf{U}_1(\mathbf{r}-\mathbf{r}_2) + \mathbf{U}_2(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) = 0, \quad \mathbf{R}_1(\mathbf{r}-\mathbf{r}_2) + \mathbf{R}_2(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) = 0, \quad \text{etc.}, \tag{5.4}$$

and in $|\mathbf{r}-\mathbf{r}_2| \leqslant a$ there are the corresponding conditions. The mean surface stress is

$\mathbf{T}_1(\mathbf{a}) + \mathbf{T}_2(\mathbf{a};\mathbf{r}_1,\mathbf{r}_2)$ at $\mathbf{r}_1 + \mathbf{a}$, on the first sphere,

$\mathbf{T}_1(\mathbf{a}) + \mathbf{T}_2(\mathbf{a};\mathbf{r}_2,\mathbf{r}_1)$ at $\mathbf{r}_2 + \mathbf{a}$, on the second sphere.

With three test spheres, centred at $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$,

$$\langle \mathbf{u}\rangle = \mathbf{U} + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_1) + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_2) + \mathbf{U}_1(\mathbf{r}-\mathbf{r}_3) + \mathbf{U}_2(\mathbf{r};\mathbf{r}_2,\mathbf{r}_3) + \mathbf{U}_2(\mathbf{r};\mathbf{r}_3,\mathbf{r}_1) + \mathbf{U}_2(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) + \mathbf{U}_3(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3),$$

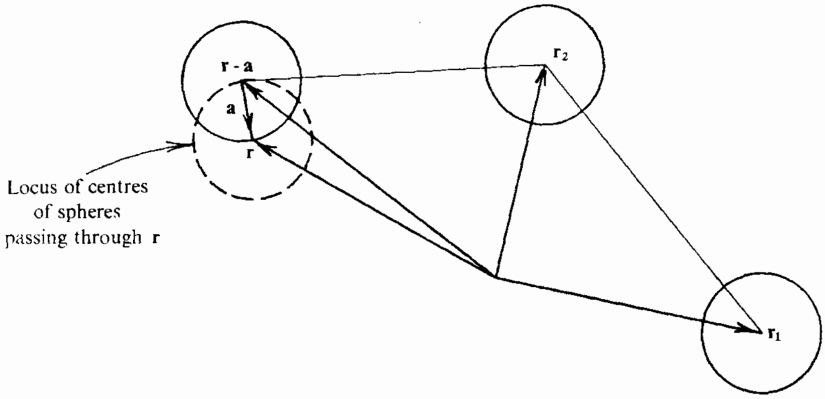

FIGURE 1. Surface of integration for resistance term.

and there are similar expressions for $\mathbf{R}$ and $\nabla\langle p\rangle$. Applying all the conditions on the quantities already defined, we obtain in $|\mathbf{r}-\mathbf{r}_1| \leqslant a$

$$\mathbf{U}_2(\mathbf{r};\mathbf{r}_2,\mathbf{r}_3)+\mathbf{U}_3(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = 0, \quad \mathbf{R}_2(\mathbf{r};\mathbf{r}_2,\mathbf{r}_3)+\mathbf{R}_3(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = 0, \quad \text{etc.,} \tag{5.5}$$

with the corresponding conditions in $|\mathbf{r}-\mathbf{r}_2| \leqslant a$ and $|\mathbf{r}-\mathbf{r}_3| \leqslant a$. The mean surface stress is $\mathbf{T}_1(\mathbf{a})+\mathbf{T}_2(\mathbf{a};\mathbf{r}_1,\mathbf{r}_2)+\mathbf{T}_2(\mathbf{a};\mathbf{r}_1,\mathbf{r}_3)+\mathbf{T}_3(\mathbf{a};\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ at $\mathbf{r}_1+\mathbf{a}$, on the first sphere, and on the others the expressions are obtained by symmetry.

Connexion between the stages is provided by the boundary conditions [(5.3), (5.4), etc.], and also by the dependence of the resistance term on the surface stress term at the following stage. Thus [see (2.2)–(2.4) and figure 1]

$$\mu\alpha^2\mathbf{U} = n\int\mathbf{T}_1(\mathbf{a})\,d\mathbf{a} = n\mathbf{F}_1, \tag{5.6}$$

$$\mathbf{R}_1(\mathbf{r}-\mathbf{r}_1) = -\int\mathbf{T}_1(\mathbf{a})\{P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1)-n\}\,d\mathbf{a}-\int\mathbf{T}_2(\mathbf{a};\mathbf{r}-\mathbf{a},\mathbf{r}_1)\,P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1)\,d\mathbf{a}, \tag{5.7}$$

$$\begin{aligned}\mathbf{R}_2(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) = &-\int\mathbf{T}_1(\mathbf{a})\{P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1,\mathbf{r}_2)-P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1)-P(\mathbf{r}-\mathbf{a}|\mathbf{r}_2)+n\}\,d\mathbf{a}\\ &-\int\mathbf{T}_2(\mathbf{a};\mathbf{r}-\mathbf{a},\mathbf{r}_1)\{P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1,\mathbf{r}_2)-P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1)\}\,d\mathbf{a}\\ &-\int\mathbf{T}_2(\mathbf{a};\mathbf{r}-\mathbf{a},\mathbf{r}_2)\{P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1,\mathbf{r}_2)-P(\mathbf{r}-\mathbf{a}|\mathbf{r}_2)\}\,d\mathbf{a}\\ &-\int\mathbf{T}_3(\mathbf{a};\mathbf{r}-\mathbf{a},\mathbf{r}_1,\mathbf{r}_2)\,P(\mathbf{r}-\mathbf{a}|\mathbf{r}_1,\mathbf{r}_2)\,d\mathbf{a}.\end{aligned} \tag{5.8}$$

In both (5.7) and (5.8), all integrals but the last represent excluded-volume effects in the neighbourhood of the test spheres.

This clearly generalizes to any number of test spheres. The equation

$$-\nabla p_m+\mu\nabla^2\mathbf{U}_m-\mu\alpha^2\mathbf{U}_m = -\mathbf{R}_m-\mu\alpha^2\mathbf{U}_m \tag{5.9}$$

holds both inside and outside the spheres, subject to the boundary conditions extended throughout the interior of the spheres:

$$\mathbf{U}_{m-1}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{k-1},\mathbf{r}_{k+1},\ldots,\mathbf{r}_m)+\mathbf{U}_m(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_m) = 0$$

$$\text{in} \quad |\mathbf{r}-\mathbf{r}_k| \leqslant a, \quad k = 1,2,\ldots,m. \tag{5.10}$$

Similar conditions hold for $\mathbf{R}_m$ and ∇p_m. Then the velocity field $\mathbf{U}_m$, everywhere continuous and without sources or sinks, satisfies

$$\mathbf{U}_m(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_m) = -\frac{1}{8\pi\mu}\sum_{k=1}^{m}\int \mathbf{T}_m(\mathbf{a};\mathbf{r}_k,\mathbf{r}_{k+1},\ldots,\mathbf{r}_m,\mathbf{r}_1,\ldots,\mathbf{r}_{k-1}) \times \mathscr{I}(\mathbf{r}-\mathbf{r}_k-\mathbf{a})\,d\mathbf{a}+\frac{1}{8\pi\mu}\int\{\mathbf{R}_m(\mathbf{r}';\mathbf{r}_1,\ldots,\mathbf{r}_m) + \mu\alpha^2\mathbf{U}_m(\mathbf{r}';\mathbf{r}_1,\ldots,\mathbf{r}_m)\}.\mathscr{I}(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'. \tag{5.11}$$

These are all deduced by induction on m, from (5.1), (5.2) and the associated boundary conditions.

Once the order of approximation m (the maximum number of spheres to be considered explicitly) has been chosen, the hierarchy of equations is truncated by the omission of $\mathbf{R}_m+\mu\alpha^2\mathbf{U}_m$. The boundary conditions (5.10), the integral equations (5.11) and the relations between $\mathbf{R}_k$ and $\mathbf{T}_{k+1}$ [(5.6)–(5.8) and the generalizations] for $k = 1,2,\ldots,m-1$ then constitute a closed system. Rather than seeking a solution to this mth-order system in isolation, it is more efficient to proceed by iteration from order 1 up to order m, so that a new correction arises at each stage. Thus, typically, the flow corrections determined at stage $m-1$ provide boundary conditions for the field problems at stage m, which begins with the m-sphere problem in the simplest effective medium, obtains the first approximations to $\mathbf{U}_m$, $\mathbf{T}_m$ and $\mathbf{R}_{m-1}$, and then successively the new corrections to $\mathbf{U}_{m-1}$, $\mathbf{T}_{m-1}$ and $\mathbf{R}_{m-2}$, and so on down to $\mathbf{T}_1$ and $\mathbf{F}_1$. The precision required at each stage naturally depends on the order of approximation intended; it decreases progressively through the iteration process.

This procedure will be illustrated in the next section, with a calculation of the mean drag correct to relative order $c^{\frac{3}{2}}$. But it is useful first to write down some exact results, referring to the first two stages of a general treatment, in which the drag calculations of § 4 are employed.

For simplicity, we take the centre $\mathbf{r}_1$ of the first test sphere to be at the origin. At the first stage, where there is one test sphere, the equations are written as

$$\left.\begin{aligned} \langle\mathbf{u}\rangle &= \mathbf{U}+\mathbf{U}_1 = \mathbf{U}+\mathbf{U}_{R1}+\mathbf{U}_{T1},\\ \text{where}\quad \mathbf{U}_{R1}(\mathbf{r}) &= \frac{1}{8\pi\mu}\int\{\mathbf{R}_1(\mathbf{r}')+\mu\alpha^2\mathbf{U}_1(\mathbf{r}')\}.\mathscr{I}(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'\\ \text{and}\quad \mathbf{U}_{T1}(\mathbf{r}) &= -\frac{1}{8\pi\mu}\int\mathbf{T}_1(\mathbf{a}).\mathscr{I}(\mathbf{r}-\mathbf{a})\,d\mathbf{a}. \end{aligned}\right\} \tag{5.12}$$

There has of course been no truncation. Then the drag calculation of (4.9)–(4.12) can be used as it stands, with $\mathbf{U}+\mathbf{U}_{R1}$ replacing $\mathbf{u}_{\text{ext}}$. For by (5.3) the integrand in the integral for $\mathbf{U}_{R1}$ vanishes inside the sphere, so that we have an externally produced flow. Also the combined flow vanishes inside the sphere. With the notation

$$B_0 = 1+\alpha a+\tfrac{1}{3}\alpha^2a^2,\quad B_2 = (\alpha^2a^2)^{-1}(e^{\alpha a}-B_0) = \tfrac{1}{6}(1+\alpha a+\tfrac{1}{4}\alpha^2a^2+\ldots), \tag{5.13}$$

the drag force is given by

$$(6\pi\mu a)^{-1}\,\mathbf{F}_1 = B_0\,\mathbf{U}+(B_0+B_2\,a^2\nabla^2)\,\mathbf{U}_{R1}(0). \tag{5.14}$$

Further, the symmetry required for (4.12) exists, and so

$$\mathbf{U}_1(\mathbf{r}) = -\tfrac{3}{4}a(B_0+B_2a^2\nabla^2)\,\mathbf{U}\,.\,\mathscr{I}(\mathbf{r})+\mathbf{U}_{R1}(\mathbf{r})$$
$$-\tfrac{3}{4}a\Big[\{(B_0+B_2a^2\nabla^2)\,\mathbf{U}_{R1}(0)\}\,.\,(\nabla\nabla-\mathbf{I}\nabla^2)\,\frac{2}{\alpha^2 r}$$
$$-\left\{e^{\alpha a}\left(1+\frac{\sinh\alpha a-\alpha a}{\alpha^3 a}\nabla^2\right)\mathbf{U}_{R1}(0)\right\}.\,(\nabla\nabla-\mathbf{I}\nabla^2)\,\frac{2}{\alpha^2 r}e^{-\alpha r}\Big]. \quad (5.15)$$

Here the first term is the reflexion of the main flow in the sphere; the second term $\mathbf{U}_{R1}$ is that directly produced by the residue $\mathbf{R}_1+\mu\alpha^2\mathbf{U}_1$ of the resistance force; the third term, in square brackets, is the reflexion of $\mathbf{U}_{R1}$ in the sphere. For the calculation in the next section, only the first two of these terms are large enough to be considered.

$\mathbf{U}_{R1}$ is now to be expressed in terms of the two-sphere stresses. With a change of variable in the integration, and the use of (5.6), (5.7) and (5.12), the result is

$$\mathbf{U}_{R1}(\mathbf{r}) = \int \mathbf{U}_{T1}(\mathbf{r}-\mathbf{r}_2)\{P(\mathbf{r}_2|0)-n\}\,d\mathbf{r}_2$$
$$-\frac{1}{8\pi\mu}\int[\{\textstyle\int\mathbf{T}_2(\mathbf{a};\mathbf{r}_2,0)\,.\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2-\mathbf{a})\,d\mathbf{a}\}P(\mathbf{r}_2|0)$$
$$-\mu\alpha^2\mathbf{U}_1(\mathbf{r}_2)\,.\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2)]\,d\mathbf{r}_2. \quad (5.16)$$

So from (5.14)

$$(6\pi\mu a)^{-1}\mathbf{F}_1 = B_0\mathbf{U}+(B_0+B_2a^2\nabla^2)\int\mathbf{U}_{T1}(\mathbf{r}-\mathbf{r}_2)\{P(\mathbf{r}_2|0)-n\}\,d\mathbf{r}_2|_{r=0}$$
$$-\frac{1}{8\pi\mu}(B_0+B_2a^2\nabla^2)\int[\{\textstyle\int\mathbf{T}_2(\mathbf{a};\mathbf{r}_2,0)\,.\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2-\mathbf{a})\,d\mathbf{a}\}P(\mathbf{r}_2|0)$$
$$-\mu\alpha^2\mathbf{U}_1(\mathbf{r}_2)\,.\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2)]\,d\mathbf{r}_2|_{r=0}. \quad (5.17)$$

In this last formula, the first term is Brinkman's result; the second represents the principal excluded-volume effect (to leading order it is the same as that found by Batchelor in the sedimentation problem). The third term contains those higher-order interactions which are introduced by the localized nature of the resistance force.

In the calculation for two spheres the drag formulae of § 4 can again be applied (though not the results of symmetry). In the neighbourhood of the first test sphere the combined flow $\mathbf{U}_1(\mathbf{r}-\mathbf{r}_2)+\mathbf{U}_2(\mathbf{r};0,\mathbf{r}_2)$ separates in the required way, and (4.11) leads to

$$(6\pi\mu a)^{-1}\mathbf{F}_2(0;\mathbf{r}_2) = (B_0+B_2a^2\nabla^2)\left\{\mathbf{U}_1(\mathbf{r}-\mathbf{r}_2)-\frac{1}{8\pi\mu}\int\mathbf{T}_2(\mathbf{a};\mathbf{r}_2,0)\right.$$
$$\left.\times\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2-\mathbf{a})\,d\mathbf{a}+\mathbf{U}_{R2}(\mathbf{r};0,\mathbf{r}_2)\right\}_{r=0}, \quad (5.18)$$

where the symbol $\mathbf{U}_{R2}$ corresponds exactly to $\mathbf{U}_{R1}$ in (5.12). This can be used to eliminate the integral in $\mathbf{T}_2$ from (5.17):

$$(6\pi\mu a)^{-1}\mathbf{F}_1 = B_0\mathbf{U}+(B_0+B_2a^2\nabla^2)$$
$$\times\left[\int\mathbf{U}_{T1}(\mathbf{r}-\mathbf{r}_2)\{P(\mathbf{r}_2|0)-n\}\,d\mathbf{r}_2+\frac{\alpha^2}{8\pi}\int\mathbf{U}_1(\mathbf{r}_2)\,.\,\mathscr{I}(\mathbf{r}-\mathbf{r}_2)\,d\mathbf{r}_2\right]_{r=0}$$
$$+\int[(6\pi\mu a)^{-1}\mathbf{F}_2(0;\mathbf{r}_2)-(B_0+B_2a^2\nabla^2)\{\mathbf{U}_1(\mathbf{r}-\mathbf{r}_2)+\mathbf{U}_{R2}(\mathbf{r};0,\mathbf{r}_2)\}]_{r=0}$$
$$\times\,P(\mathbf{r}_2|0)\,d\mathbf{r}_2. \quad (5.19)$$

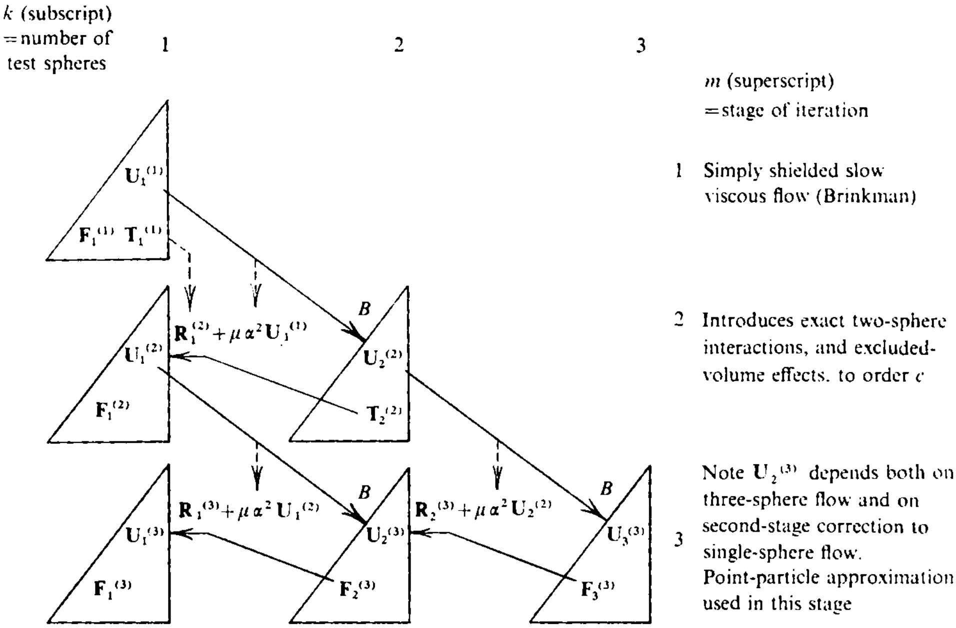

FIGURE 2. Iteration scheme for mean drag calculation (can be extended to any order).

Because this involves only the additional drag $\mathbf{F}_2$, it is a convenient form for introducing the exact two-sphere interactions. But (5.17) is found to be more suitable when it comes to the three-sphere effects in § 6.

6. Calculation of the drag on a sphere, as far as the term in $c^{\frac{3}{2}}$

The intended accuracy requires that the calculation be taken to the stage of three test spheres. It is hoped in this way to illustrate the general process, to confirm the correctness of the term in c and also to obtain an indication of the usefulness of the expansion at practical values of c.

In what follows, subscripts refer to the number of test spheres, superscripts to the stage of iteration. The first stage ($\mathbf{U}_1^{(1)}$, $\mathbf{F}_1^{(1)}$) is simply Brinkman's solution for viscous flow with simple shielding past one fixed sphere. At the second stage, $\mathbf{U}_1^{(1)}$ provides the boundary condition for the difference flow $\mathbf{U}_2^{(2)}$ when there are two fixed spheres with simple shielding; the corresponding difference stress $\mathbf{T}_2^{(2)}$ leads to the first correction $\mathbf{R}_1^{(2)}$ to the resistance in the single-sphere problem, and hence to the corrections $\mathbf{U}_1^{(2)}$ and $\mathbf{F}_1^{(2)}$, for velocity and drag. At the third stage (and here the point-particle approximation is sufficient) $\mathbf{U}_2^{(2)}$ provides the boundary condition for $\mathbf{U}_3^{(3)}$, the three-sphere difference flow with simple shielding; proceeding via the two-sphere problem we find the second correction $\mathbf{F}_1^{(3)}$ to the required drag. The scheme is set out in figure 2; the final approximation for the mean drag is $\mathbf{F}_1^{(1)}+\mathbf{F}_1^{(2)}+\mathbf{F}_1^{(3)}$.

For stage m, the triangle in the kth column denotes a field problem with k spheres. It depends on the $(k-1)$-sphere problem at stage $m-1$, through the boundary conditions (5.3)–(5.5), as shown by the arrows marked B. It depends on the $(k+1)$-sphere problem at stage m, through the resistance terms (5.7) and (5.8), as shown by the arrowheads under the symbols $\mathbf{R}_k^{(m)}$. It depends on the k-sphere problem at stage $m-1$, as shown by the dotted arrows, principally through the term $\mu\alpha^2\mathbf{U}_k^{(m-1)}$ added to the resistance, but also at one point through the excluded-volume effect on the resistance (5.7). For the accuracy aimed at here, the stress distribution $\mathbf{T}$ over the spheres is required in only one of the field problems. Apart from this, the overall force is sufficient, as shown by the entry $\mathbf{F}$.

It may be noted in general that each use of a resistance residue term introduces a factor of order c and an extra volume integration, so that at the $(m+1)$th stage there is a factor c^m times an m-fold volume integral. The possible order of magnitude of the result will depend on how many times, and with what argument, the Green's function occurs as a factor in the integrand.

Stage 1. The solution is known from (5.13)–(5.15), when $\mathbf{U}_{R1}$ is omitted:

$$\left.\begin{aligned} \mathbf{U}_1^{(1)} &= \begin{cases} -\mathbf{U} & \text{in} \quad r \leqslant a, \\ -\tfrac{3}{4}a(B_0+B_2a^2\nabla^2)\,\mathbf{U}\,.\,\mathscr{I}(\mathbf{r}) & \text{in} \quad r > a, \end{cases} \\ \mathbf{U}_e^{(1)} &= (6\pi\mu a)^{-1}\,\mathbf{F}_1^{(1)} = B_0\mathbf{U} = (1+\alpha a+\tfrac{1}{3}\alpha^2a^2)\,\mathbf{U}, \end{aligned}\right\} \tag{6.1}$$

and from (5.12), $\mathbf{U}_{T_1}^{(1)} = \mathbf{U}_1^{(1)}$. With the self-consistency equation (2.5) for α, the first iteration gives

$$\alpha^2a^2 = \alpha_0^2a^2\{1+\alpha_0a+O(\alpha_0^2a^2)\}, \quad \alpha_0^2a^2 = \tfrac{9}{2}c. \tag{6.2}$$

Stage 2. We introduce a tensor coefficient $\mathbf{D}$ for the total drag on each of two fixed spheres in uniform flow, with simple shielding:

$$\mathbf{F}_1^{(1)}+\mathbf{F}_2^{(2)} = 6\pi\mu a\mathbf{U}\,.\,\mathbf{D}(\mathbf{r},\alpha),$$

so that

$$\mathbf{F}_2^{(2)} = 6\pi\mu a\mathbf{U}\,.\,(\mathbf{D}-\mathbf{I}B_0).$$

$\mathbf{D}$ is known for $\alpha = 0$ (Happel & Brenner 1965, p. 269: in their notation it is $T_1\hat{\mathbf{r}}\hat{\mathbf{r}}+T_2(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}})$; see also Goldman, Cox & Brenner 1966). And it is shown in appendix B that

$$\mathbf{D}(\mathbf{r},\alpha) = \mathbf{D}(\mathbf{r},0)\,.\,\{\mathbf{I}+2\alpha a\mathbf{D}(\mathbf{r},0)\}+O(\alpha^2ar).$$

Then the correction to the effective velocity resulting from stage 2 can be written down from the exact expression (5.19), in which $\mathbf{U}_{R2}$ is to be omitted, and the first approximations used for all the other velocities that appear:

$$\mathbf{U}_e^{(2)} = \mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3+\mathbf{W}_4,$$

where

$$\begin{aligned} \mathbf{W}_1 &= (B_0+B_2a^2\nabla_{\mathbf{r}'}^2)\int \mathbf{U}_1^{(1)}(\mathbf{r}'-\mathbf{r})\,\{P(\mathbf{r}|0)-n\}\,d\mathbf{r}\big|_{\mathbf{r}'=0}, \\ \mathbf{W}_2 &= (B_0+B_2a^2\nabla_{\mathbf{r}'}^2)\int \frac{\alpha^2}{8\pi}\,\mathbf{U}_1^{(1)}(\mathbf{r}'-\mathbf{r})\,.\,\mathscr{I}(\mathbf{r})\,d\mathbf{r}\bigg|_{\mathbf{r}'=0}, \\ \mathbf{W}_3 &= (\tfrac{3}{4}a)^2B_0^3\,\mathbf{U}\,.\int \mathscr{I}^2(\mathbf{r})\,.\,\{\mathbf{I}+\tfrac{3}{4}aB_0\mathscr{I}(\mathbf{r})\}^{-1}P(\mathbf{r}|0)\,d\mathbf{r}, \\ \mathbf{W}_4 &= \mathbf{U}\,.\int [\mathbf{D}(\mathbf{r},\alpha)-B_0\{\mathbf{I}+\tfrac{3}{4}aB_0\mathscr{I}(\mathbf{r})\}^{-1} \\ &\qquad +\tfrac{3}{4}aB_2(2B_0+B_2a^2\nabla^2)\,a^2\nabla^2\mathscr{I}(\mathbf{r})]\,P(\mathbf{r}|0)\,d\mathbf{r}. \end{aligned}$$

The integral containing $\mathbf{F}_2$ in (5.19) has been split into two parts $\mathbf{W}_3$ and $\mathbf{W}_4$. The former contains long-range interactions; it can be evaluated as a series, with terms in $c(\alpha a)^{-1}$, $c \log \alpha a$, $c\alpha a$ etc., and the first of these is cancelled by $\mathbf{W}_2$. The integral $\mathbf{W}_4$ must be calculated numerically from the exact results for $\mathbf{D}$, by the same procedure as described by Batchelor (1972, p. 261). To leading order ($\alpha = 0$) there are only short-range interactions, and there is no complication. But for the term in $c\alpha a$, a further expression involving Green's functions must be removed and evaluated analytically: accounting for all those interactions that vary as r^{-4} out to the shielding radius. What remains in the integrand then has the expansion

$$c\mathbf{G}_1(\mathbf{r}) + c\alpha a \mathbf{G}_2(\mathbf{r}) + O(c\alpha^2 a^2 r^{-3} U), \quad a < r < \alpha^{-1},$$
$$O(ca^2 r^{-5} U), \quad \alpha^{-1} < r,$$

where $\mathbf{G}_1 = O(a^2 r^{-5} U)$ and $\mathbf{G}_2 = O(ar^{-4} U)$. The integral is then

$$c\int_{r>2a} \mathbf{G}_1(\mathbf{r})\, d\mathbf{r} + c\alpha a \int_{r>2a} \mathbf{G}_2(\mathbf{r})\, d\mathbf{r} + O(Uc^2 \log c).$$

In evaluating $\mathbf{W}_1$ and $\mathbf{W}_2$, it is most convenient to let the Laplacian operate on $\mathbf{U}_1^{(1)}$ inside the integral. This can be done, as a limit process, if the velocity field is modified slightly in a thin shell containing the sphere surface. In the limit the integral of $\mu\nabla^2\mathbf{U}_1^{(1)}$ over the thin shell is just the total shear drag $\mathbf{F}_{1s}^{(1)}$. When the simplest probability densities are used (§2, second paragraph), the various contributions to the effective velocity have the values

$$\left.\begin{aligned}
B_0\mathbf{U} &= \mathbf{U}(1 + \alpha a + \tfrac{1}{3}\alpha^2 a^2),\\
\mathbf{W}_1 &= \mathbf{U}c(5 + 2\alpha a + \tfrac{1}{3}\alpha^2 a^2),\\
\mathbf{W}_2 &= -\mathbf{U}(\tfrac{1}{2}\alpha a + \tfrac{1}{3}\alpha^2 a^2),\\
\mathbf{W}_3 &= \mathbf{U}c\{9/4\alpha a + \tfrac{135}{64}(\log \tfrac{2}{9}\alpha^2 a^2 + 6{\cdot}7056\ldots) + \tfrac{135}{8}\alpha a(\log \tfrac{2}{9}\alpha^2 a^2 + 5{\cdot}0579\ldots)\\
&\qquad + O(\alpha^2 a^2 \log \alpha a)\},\\
\mathbf{W}_4 &= \mathbf{U}c\{1{\cdot}120\ldots + 8{\cdot}583\ldots\alpha a + O(\alpha^2 a^2 \log \alpha a)\},
\end{aligned}\right\} \quad (6.3)$$

and the total to this stage is

$$\mathbf{U}_e^{(1)} + \mathbf{U}_e^{(2)} = \mathbf{U}\{1 + \tfrac{1}{2}\alpha a + 9c/4\alpha a + \tfrac{135}{64}c(\log \tfrac{2}{9}\alpha^2 a^2 + 9.608\ldots)$$
$$+ \tfrac{135}{8}c\alpha a(\log \tfrac{2}{9}\alpha^2 a^2 + 5{\cdot}566\ldots) + O(c\alpha^2 a^2 \log \alpha a)\}.$$

The elimination of α, see (2.5), now requires just one use of (6.2), because the combination $\frac{1}{2}\alpha a + 9c/(4\alpha a)$ depends quadratically on the deviation of αa from its leading approximation. The result is

$$\mathbf{U}_e^{(1)} + \mathbf{U}_e^{(2)} = \mathbf{U}\left\{1 + \frac{3}{\sqrt{2}}c^{\frac{1}{2}} + \tfrac{135}{64}c(\log c + 9{\cdot}608) + \frac{405}{8\sqrt{2}}c^{\frac{3}{2}}(\log c + 5{\cdot}725) + \ldots\right\}. \quad (6.4)$$

The most unsatisfactory part of the calculation so far seems to be the expansion of the integral $\mathbf{W}_3$ as in (6.3), for small values of αa. Numerical integration shows that when c is larger than 0·002 the first term alone is a better approximation than the first three terms. Clearly the $c \log c$ term is of mainly theoretical interest; it is most unlikely that it could be verified by experiment.

As an alternative, we can write $\mathbf{W}_3$ as $\mathbf{U}cG$, where G is known numerically as a function of αa, and omit $\mathbf{W}_4$ entirely in this exploratory approach (it is generally small compared with $\mathbf{W}_3$ in the range of c for which (6.3) can be trusted). The self-consistency equation then takes the form

$$9c^2\{G(\alpha a)+5+2\alpha a+\tfrac{1}{3}\alpha^2a^2\}+9c(1+\tfrac{1}{2}\alpha a) = 2\alpha^2a^2,$$

so that c, and hence the mean drag, can be found for each value of αa.

Figure 5 shows the resulting mean drag coefficient for small values of c, together with that given by Brinkman's theory. It can be seen that the two-sphere interactions produce a small reduction in the drag, at concentrations less than 0.014.

In figure 7 the graphs are extended to concentrations that are not small, and the shielding radius is shown as well. The reasonable appearance of the results (see the comparison made by Tam (1969) between experimental data and Brinkman's theory; and compare the value of c at close packing, about 0·74, with those at which infinite drag would be predicted by the theories: 0·79 for the present theory and 0·67 for Brinkman's) raises the question whether the flow equations with simple shielding may be a suitable first approximation even in the case of densely packed spheres. Actually the flow equations then reduce to Darcy's law with a skin effect near the boundaries,† and it would not be too difficult to correct for the omission of $\mathbf{W}_4$. But the multivariate distribution of spheres in close proximity would presumably play a bigger role.

Stage 3. In the three-sphere problem we are interested only in the leading contributions, supposing initially that these come from large values of the separations $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_2-\mathbf{r}_3$ ($\mathbf{r}_1$ being taken to be at the origin), for which the point-particle approximation is sufficient. Thus (for example) in the integral of $\mathbf{T}(\mathbf{a}).\mathcal{I}(\mathbf{r}_2-\mathbf{a})$ over the sphere, the argument $\mathbf{r}_2-\mathbf{a}$ is replaced by $\mathbf{r}_2$. The resistance force is just $-n$ times the drag force on an extra test particle. The drag expression (4.11) is replaced by $-\mu\alpha^2n^{-1}\mathbf{u}_{\text{ext}}(0)$, as if the externally produced velocity field were uniform, but using the exact drag coefficient for that case. It is necessary to retain at least the correction factor $1+\alpha a$ to the simple Stokes drag, because of the cancelling of the leading terms in the resistance residues $\mathbf{R}_1+\mu\alpha^2\mathbf{U}_1$, etc.

To this approximation,

$$\mathbf{U}_1^{(1)} = (-\alpha^2/8\pi n)\,\mathbf{U}.\mathcal{I}(\mathbf{r}). \tag{6.5}$$

The two-particle problem, with the use of (5.17) and the symmetry between the two particles, now requires just a matrix inversion:

$$\left.\begin{aligned}\mathbf{F}_2^{(2)}(0;\mathbf{r}_2) = \mathbf{F}_2^{(2)}(\mathbf{r}_2;0) &= -\frac{\mu\alpha^4}{8\pi n^2}\mathbf{U}.\mathcal{I}(\mathbf{r}_2).\left\{\mathbf{I}+\frac{\alpha^2}{8\pi n}\mathcal{I}(\mathbf{r}_2)\right\}^{-1}\\ \text{and}\quad \mathbf{U}_2^{(2)}(\mathbf{r};0,\mathbf{r}_2) &= \left(\frac{\alpha^2}{8\pi n}\right)^2\mathbf{U}.\mathcal{I}(\mathbf{r}_2).\left\{\mathbf{I}+\frac{\alpha^2}{8\pi n}\mathcal{I}(\mathbf{r}_2)\right\}^{-1}.\{\mathcal{I}(\mathbf{r})+\mathcal{I}(\mathbf{r}-\mathbf{r}_2)\}.\end{aligned}\right\} \tag{6.6}$$

For the three-particle system, we write

$$\left.\begin{aligned}&\mathbf{F}_3^{(3)}(0;\mathbf{r}_2,\mathbf{r}_3) = \frac{\mu\alpha^2}{n}\mathbf{X},\quad \mathbf{F}_3^{(3)}(\mathbf{r}_2;\mathbf{r}_3,0) = \frac{\mu\alpha^2}{n}\mathbf{Y},\quad \mathbf{F}_3^{(3)}(\mathbf{r}_3;0,\mathbf{r}_2) = \frac{\mu\alpha^2}{n}\mathbf{Z},\\ &\frac{\alpha^2}{8\pi n}\mathcal{I}(\mathbf{r}_3-\mathbf{r}_2) = \mathbf{A},\quad \frac{\alpha^2}{8\pi n}\mathcal{I}(\mathbf{r}_3) = \mathbf{B},\quad \frac{\alpha^2}{8\pi n}\mathcal{I}(\mathbf{r}_2) = \mathbf{C};\end{aligned}\right\} \tag{6.7}$$

† Cf. Saffman (1971).

then with (6.6) and two similar expressions, the equations obtained are

$$\left.\begin{aligned} \mathbf{X}+\mathbf{Y}.\mathbf{C}+\mathbf{Z}.\mathbf{B} &= \mathbf{U}.\mathbf{A}.(\mathbf{I}+\mathbf{A})^{-1}.(\mathbf{B}+\mathbf{C}),\\ \mathbf{X}.\mathbf{C}+\mathbf{Y}+\mathbf{Z}.\mathbf{A} &= \mathbf{U}.\mathbf{B}.(\mathbf{I}+\mathbf{B})^{-1}.(\mathbf{C}+\mathbf{A}),\\ \mathbf{X}.\mathbf{B}+\mathbf{Y}.\mathbf{A}+\mathbf{Z} &= \mathbf{U}.\mathbf{C}.(\mathbf{I}+\mathbf{C})^{-1}.(\mathbf{A}+\mathbf{B}).\end{aligned}\right\} \tag{6.8}$$

The quantity required is now the resistance residue

$$\begin{aligned} \mathbf{R}_2^{(3)}(\mathbf{r}_3;0,\mathbf{r}_2)+\mu\alpha^2\mathbf{U}_2^{(2)}(\mathbf{r}_3;0,\mathbf{r}_2) &= -n\mathbf{F}_3^{(3)}(\mathbf{r}_3;0,\mathbf{r}_2)+\mu\alpha^2\mathbf{U}_2^{(2)}(\mathbf{r}_3;0,\mathbf{r}_2)\\ &= \mu\alpha^2\{-\mathbf{Z}+\mathbf{U}.\mathbf{C}.(\mathbf{I}+\mathbf{C})^{-1}.(\mathbf{A}+\mathbf{B})\}\\ &= \mu\alpha^2(\mathbf{X}.\mathbf{B}+\mathbf{Y}.\mathbf{A}),\end{aligned}$$

and the flow that results directly from this is

$$\mathbf{U}_{R2} = \frac{\alpha^2}{8\pi}\int(\mathbf{X}.\mathbf{B}+\mathbf{Y}.\mathbf{A}).\mathscr{I}(\mathbf{r}-\mathbf{r}_3)\,d\mathbf{r}_3. \tag{6.9}$$

The correction to the boundary condition for the two-sphere problem is also needed (by symmetry the same value applies at $\mathbf{r}_2$ and at O):

$$\begin{aligned} \mathbf{U}_1^{(2)}(-\mathbf{r}_2) &= \mathbf{U}_{R1}(-\mathbf{r}_2)\\ &= -\frac{1}{8\pi\mu}\int\{n\mathbf{F}_2^{(2)}(\mathbf{r}_3;\mathbf{r}_2)-\mu\alpha^2\mathbf{U}_1^{(1)}(\mathbf{r}_3-\mathbf{r}_2)\}.\mathscr{I}(\mathbf{r}_3)\,d\mathbf{r}_3\\ &= -n\mathbf{U}.\int\mathbf{A}^2.(\mathbf{I}+\mathbf{A})^{-1}.\mathbf{B}\,d\mathbf{r}_3. \end{aligned}\tag{6.10}$$

The drag on the sphere at $\mathbf{r}_2$ due to the combination of (6.9) and (6.10) leads to the resistance residue

$$\begin{aligned} \mathbf{R}_1^{(3)}(\mathbf{r}_2)+\mu\alpha^2\mathbf{U}_1^{(2)}(\mathbf{r}_2) &= -\mu\alpha^2[\{\mathbf{U}_1^{(2)}(\mathbf{r}_2)+\mathbf{U}_{R2}(\mathbf{r}_2)\}.(\mathbf{I}+\mathbf{C})^{-1}-\mathbf{U}_1^{(2)}(\mathbf{r}_2)]\\ &= -\mu\alpha^2 n\int\{(\mathbf{X}.\mathbf{B}+\mathbf{Y}.\mathbf{A}).\mathbf{A}+\mathbf{U}.\mathbf{A}^2.(\mathbf{I}+\mathbf{A})^{-1}.\mathbf{B}.\mathbf{C}\}\\ &\quad\times(\mathbf{I}+\mathbf{C})^{-1}d\mathbf{r}_3,\end{aligned}$$

and from the resulting flow, the final correction to the equivalent velocity felt by the sphere at O is

$$\mathbf{U}_e^{(3)} = -\iint\limits_{\substack{r_2,r_3>2a\\|\mathbf{r}_2-\mathbf{r}_3|>2a}} n^2\{\mathbf{X}.\mathbf{B}.\mathbf{A}+\mathbf{Y}.\mathbf{A}^2+\mathbf{U}.\mathbf{A}^2.(\mathbf{I}+\mathbf{A})^{-1}.\mathbf{B}.\mathbf{C}\}\times(\mathbf{I}+\mathbf{C})^{-1}.\mathbf{C}\,d\mathbf{r}_2 d\mathbf{r}_3. \tag{6.11}$$

The solution of (6.8) for $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ and hence the integrand in (6.11) can be expressed as power series in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. It is now necessary to examine the orders of magnitude of the various contributions to the integral. First, consider the terms of lowest degree (the fifth) in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Their contribution to (6.11) is of order $Uc^2(\alpha a)^{-1}$, or $Uc^2(\alpha a)^{-1}\log\alpha a$ for the term $\mathbf{B}.\mathbf{A}^3.\mathbf{C}$, which is of third degree in one of the three factors. This last term is actually to be grouped with a second class of terms: all those having combined degree two in two of the factors (thus $\mathbf{B}.\mathbf{A}^3.\mathbf{C}, \mathbf{C}.\mathbf{A}^4.\mathbf{C}, \mathbf{B}.\mathbf{A}^5.\mathbf{C}, \ldots$, which come from solving the second and third equations of (6.8) simultaneously for $\mathbf{Y}$ and $\mathbf{Z}$). Noting (from figure 3) that $\mathbf{A}$ depends on $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{B}$ on $\mathbf{r}_3$ and $\mathbf{r}_1$, and $\mathbf{C}$ on $\mathbf{r}_1$ and $\mathbf{r}_2$, we see that each of these terms decays approximately (as far as the shielding radius) as the inverse square of the separation of the first particle from the other two.

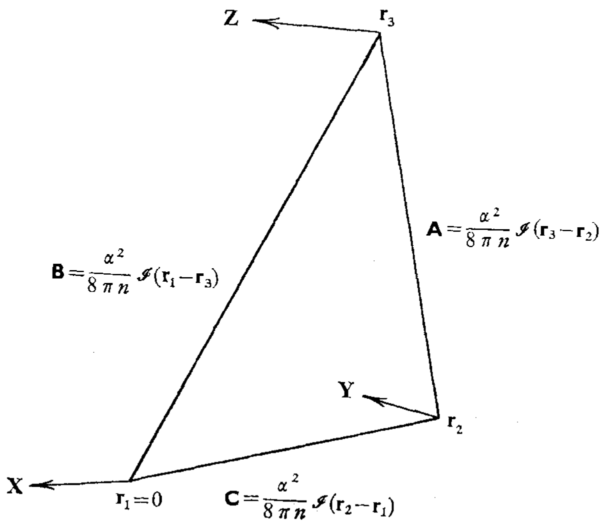

FIGURE 3. Tensor coefficients for three-sphere problem.

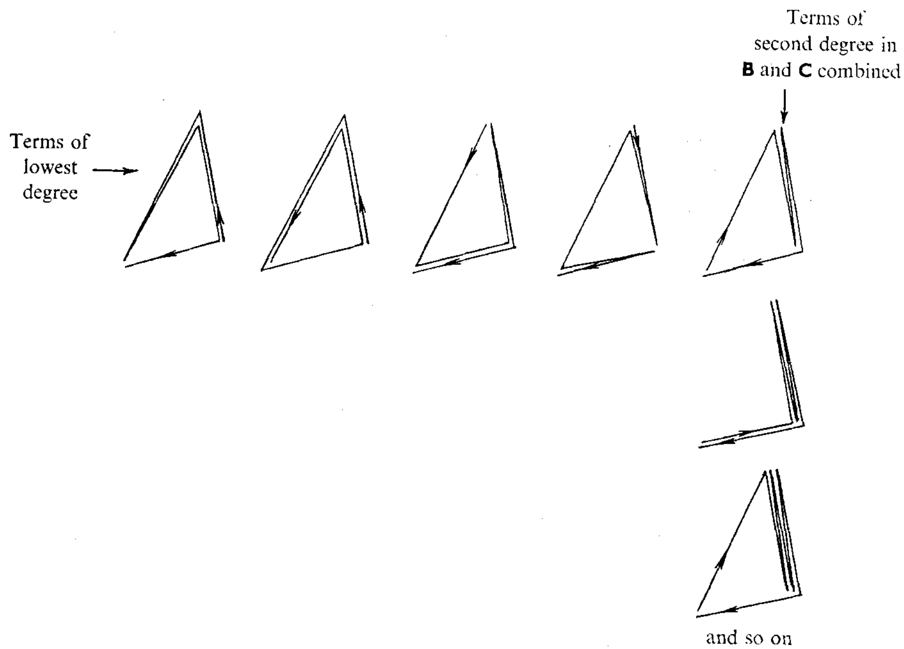

FIGURE 4. Interaction diagrams.

It follows that, after $\mathbf{B}.\mathbf{A}^3.\mathbf{C}$, the contributions of these terms to the integral are all of order $Uc^2(\alpha a)^{-1}$. All other terms have integrals of order $U(c\log c)^2$ or smaller, and can be omitted here. Interaction diagrams for the significant terms are shown in figure 4.

The resulting effective velocity correction is

$$\mathbf{U}_e^{(3)} = -n^2\mathbf{U}.\iint_{\substack{r_2, r_3 > 2a \\ |\mathbf{r}_2 - \mathbf{r}_3| > 2a}} \{\mathbf{A}.\mathbf{B}^2.\mathbf{A} + \mathbf{A}.\mathbf{C}.\mathbf{B}.\mathbf{A} + \mathbf{B}.\mathbf{C}.\mathbf{A}^2 + \mathbf{A}^2.\mathbf{B}.\mathbf{C} + \mathbf{B}.\mathbf{A}^3 \times (\mathbf{I} - \mathbf{A}^2)^{-1} - \mathbf{C}.\mathbf{A}^4.(\mathbf{I} - \mathbf{A}^2)^{-1}\}.\mathbf{C}\,d\mathbf{r}_2\,d\mathbf{r}_3,$$

which becomes, after some rearrangement, and omission of all but the esssential restrictions on the region of integration,

$$\mathbf{U}_e^{(3)} = -n^2\mathbf{U}.\iint\{\mathbf{A}.\mathbf{B}^2.\mathbf{A}+\mathbf{A}.\mathbf{C}.\mathbf{B}.\mathbf{A}+\mathbf{B}.\mathbf{C}.\mathbf{A}^2+\mathbf{A}^2.\mathbf{B}.\mathbf{C}+(\mathbf{B}-\mathbf{C}).\mathbf{A}^3\} \times \mathbf{C}\,d\mathbf{r}_2\,d\mathbf{r}_3 - n^2\mathbf{U}.\int \mathbf{C}.\left\{\int_{|\mathbf{r}_3-\mathbf{r}_2|>2a} \mathbf{A}^3.(\mathbf{I}+\mathbf{A})^{-1}d\mathbf{r}_3\right\}.\mathbf{C}\,d\mathbf{r}_2. \quad (6.12)$$

The integral of $(\mathbf{B}-\mathbf{C}).\mathbf{A}^5.(\mathbf{I}-\mathbf{A}^2)^{-1}.\mathbf{C}$ has also been omitted as insignificant.

The form of the second integral in (6.12) shows that it represents a correction to the interaction between two spheres when the drag on the second is modified by the presence of a third. So it is clear that the point-particle approximation will not be sufficient here, but that the integral of $\mathbf{A}^3(\mathbf{I}+\mathbf{A})^{-1}$ should include all the drag contributions from stage 2. Thus corrected, the second term in (6.12) becomes

$$\tfrac{1}{2}\alpha a\mathbf{U}\{9c/4\alpha a - \tfrac{1}{2}\alpha a - \tfrac{1}{3}\alpha^2a^2 + \tfrac{135}{64}c(\log c + 9{\cdot}608\ldots)\} = \tfrac{135}{128}\mathbf{U}c\alpha a(\log c + 7{\cdot}830\ldots) + O(Uc^2\log^2 c).$$

In the other part of (6.12) the short-range interactions are not significant for the $c^{\frac{3}{2}}$ term, and it follows that the point-particle approximation is sufficient there. (That is, if the power index of dependence of the integrand on any one of the separations $\mathbf{r}_2$, $\mathbf{r}_3-\mathbf{r}_2$ and $\mathbf{r}_3$, as far as the shielding radius, is decreased by unity algebraically, the integral ceases to be significant.) The total correction from stage 3 is finally found to be

$$\mathbf{U}_e^{(3)} = (405/128\sqrt{2})\,\mathbf{U}c^{\frac{3}{2}}(\log c + 0{\cdot}379) + O(Uc^2\log^2 c).$$

When this is combined with (6.4), the coefficients of c and $c^{\frac{3}{2}}\log c$ differ from those given by Childress (1972), following his equation (8.1). Dr Childress says that he has now discovered errors in his formula (1.2c), where $-\frac{76}{15}$ should be $+\frac{1}{10}$, and in the fourth line after (8.1), where the value he gives for D_{31} should be multiplied by five. But there remain small discrepancies to be resolved.

7. Drag in uniform longitudinal and transverse flow through arrays of infinite parallel rods

The work of the preceding sections can to a great extent be taken over for the two-dimensional situations. Position vectors, denoted as before by $\mathbf{r}, \mathbf{r}_1, \ldots$, now lie in a plane; the rods are circular cylinders perpendicular to the plane, distributed with mean number density λ per unit area of the plane. Then λ is also the mean rod length per unit volume, and the volume fraction is $c = \pi\lambda a^2$. The overall mean velocities are denoted by U and $\mathbf{U}$ respectively in the longitudinal and transverse flow problems. The mean drag per unit length of rod and mean resistance per unit volume are written as follows.

In longitudinal flow

$$F = (2\pi\mu/M_l)\,U, \quad (7.1)$$

$$-R = \lambda F = \mu\alpha_l^2 U, \quad \text{so that} \quad \alpha_l^2 a^2 = 2c/M_l. \quad (7.2)$$

In transverse flow

$$\mathbf{F} = (4\pi\mu/M_t)\,\mathbf{U}, \tag{7.3}$$

$$-\mathbf{R} = \lambda\mathbf{F} = \mu\alpha_t^2\mathbf{U}, \quad \text{so that} \quad \alpha_t^2 a^2 = 4c/M_t. \tag{7.4}$$

The last part of § 4 gives the appropriate Green's functions and drag formulae. The schemes set out in §§ 5 and 6 apply, and even the integrals expressing the results in § 6 can be used with a few modifications. But the relative orders of magnitude of the contributions are different: there is now an infinite series of negative powers of $\log c$, corresponding to interactions between increasing numbers of cylinders, but the effects of finite radius still appear as an order-c term. All the same, the calculations will be taken as far as order c in the earlier stages, because it is of interest to present the results numerically over a wide range of c. The case of longitudinal flow, which has been worked out more fully, is presented first.

Longitudinal flow

Stage 1

$$U_1^{(1)} = \begin{cases} -U, & r \leqslant a, \\ -U\,K_0(\alpha_l r)/K_0(\alpha_l a), & r > a. \end{cases}$$

In combination with (7.1) the drag result can be written as

$$\frac{U}{M_{l1}} = \frac{F_1^{(1)}}{2\pi\mu} = \frac{U\alpha_l a K_1(\alpha_l a)}{K_0(\alpha_l a)},$$

and from (7.2) the self-consistency equation is, to a sufficient approximation for small c,

$$M_{l1} - \tfrac{1}{2}\log M_{l1} + \gamma = \tfrac{1}{2}\log(2/c). \tag{7.5}$$

Euler's constant γ arises in the expansion of K_0 about the origin.

Stage 2. It is convenient to introduce the notation

$$M_{l0} = K_0(\alpha_l a). \tag{7.6}$$

As in § 6,

$$U/M_{l2} = F_1^{(1)}/2\pi\mu + W_1 + W_2 + W_3 + W_4, \tag{7.7}$$

where
$$W_1 = \frac{1}{M_{l0}}\int U_1^{(1)}\{P(\mathbf{r}|0)-\lambda\}\,d\mathbf{r} = \frac{Uc}{M_{l0}}\left\{4 + \frac{1{\cdot}5 - 4\log 2}{M_{l0}} + O(c)\right\},$$

$$\begin{aligned} W_2 + W_3 &= \frac{1}{M_{l0}}\int \frac{\alpha_l^2}{2\pi} U_1^{(1)}(\mathbf{r})\,K_0(\alpha_l r)\,d\mathbf{r} + \frac{U}{M_{l0}^2}\int \frac{K_0^2(\alpha_l r)}{M_{l0}+K_0(\alpha_l r)} P(\mathbf{r}|0)\,d\mathbf{r} \\ &= \frac{M_l - M_{l0}}{2M_{l0}^3}U + O(Uc) - \frac{M_l}{M_{l0}^3}U\int_{2\alpha_l a}^{\infty} \frac{xK_0^3(x)}{M_{l0}+K_0(x)}\,dx, \end{aligned} \tag{7.8}$$

$$W_4 = U\int\left\{D_l(\mathbf{r},\alpha_l) - \frac{1}{M_{l0}+K_0(\alpha_l a)}\right\}P(\mathbf{r}|0)\,d\mathbf{r}.$$

The calculation of W_4, which like W_1 is of order Uc, is described briefly in appendix B. The first term in (7.8), of order $U(\log c)^{-4}$, results from the iteration process, in which inexact drag coefficients are used for the interactions at intermediate stages.

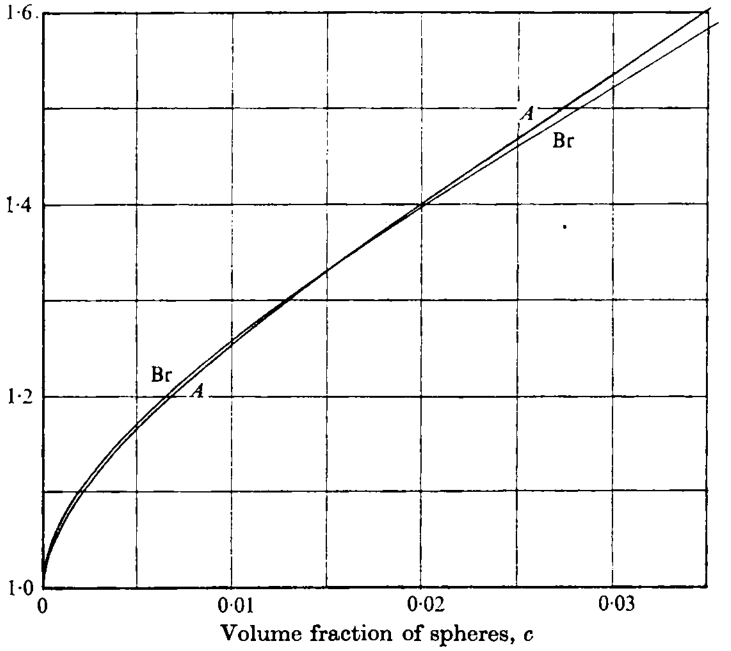

FIGURE 5. Mean drag coefficient $F/6\pi\mu aU$ for spheres. A, second-order theory (§6, stage 2); Br, Brinkman's theory.

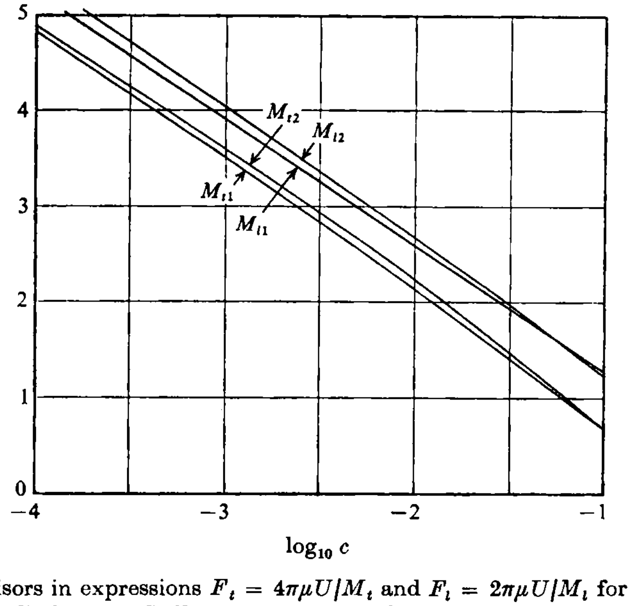

FIGURE 6. Divisors in expressions $F_t = 4\pi\mu U/M_t$ and $F_l = 2\pi\mu U/M_l$ for mean drag per unit length of cylinder (§7). Suffixes: t, transverse flow; l, longitudinal flow; 1, first-order theory; 2, second-order theory.

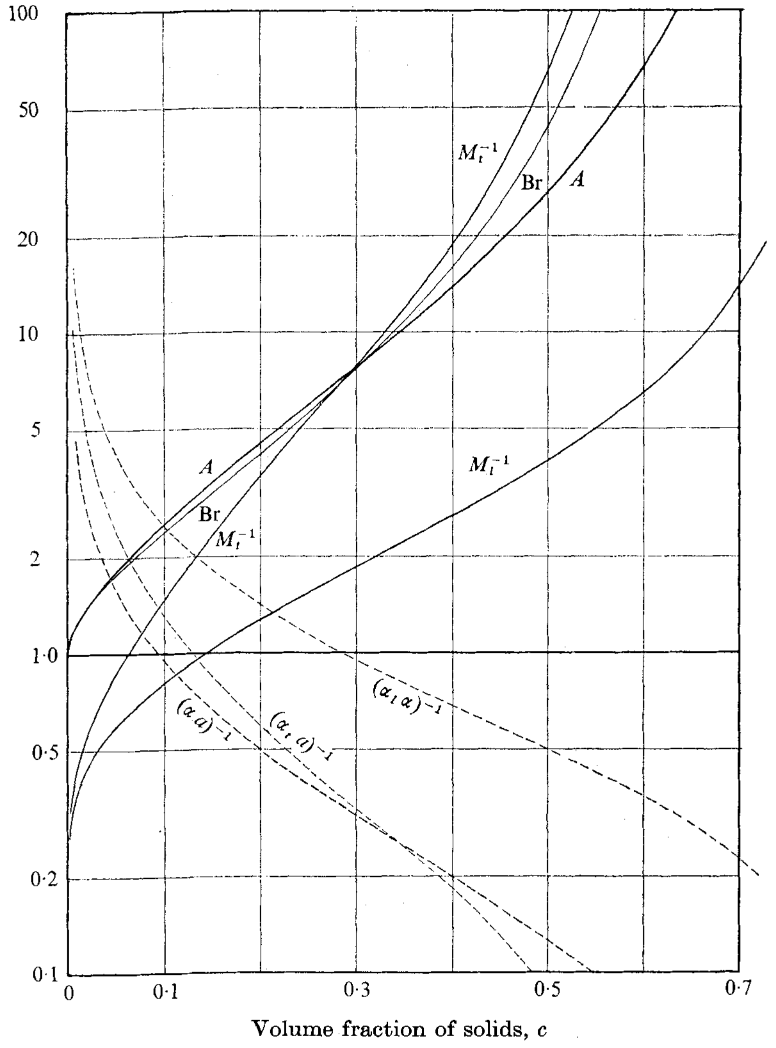

FIGURE 7. Mean drag coefficients and shielding radii according to the second-order theory extended to large concentrations (§6, end of stage 2; §7). ——, drag coefficients. Spheres: A, $F/6\pi\mu a U$; Br, Brinkman's theory. Cylinders: M_t^{-1}, $F/4\pi\mu U$, transverse; M_l^{-1}, $F/2\pi\mu U$, longitudinal. – – –, corresponding shielding radii as multiples of a; $(\alpha a)^{-1}$, $(\alpha_t a)^{-1}$, $(\alpha_l a)^{-1}$.

The important contribution from this stage is the integral that appears at the end of expression (7.8); its asymptotic form for large M_{l0} is

$$\frac{M_l}{M_{l0}^3} U \left(-\frac{0{\cdot}586}{M_{l0}} + \frac{1{\cdot}052}{M_{l0}^2} - \dots \right). \tag{7.9}$$

Then the drag equation (7.7) becomes

$$\frac{1}{M_{l2}} = \frac{1}{M_{l0}} + \frac{1}{M_{l0}^3} \left\{ \tfrac{1}{2}(M_l - M_{l0}) - 0{\cdot}586 \frac{M_l}{M_{l0}} + 1{\cdot}052 \frac{M_l}{M_{l0}^2} + O(M_{l0}^{-2}) \right\},$$

which yields, after the use of (7.6) and (7.2), the self-consistency equation

$$M_{l2} - \tfrac{1}{2}\log M_{l2} + \gamma - \frac{0{\cdot}586}{M_{l2}} + \frac{1{\cdot}345}{M_{l2}^2} + O(M_{l2}^{-3}) = \tfrac{1}{2}\log\frac{2}{c}. \qquad (7.10)$$

Stage 3. The treatment of this stage in § 6 needs only the obvious modifications, and the discussion of orders of magnitude after (6.11) is readily adapted to select the group of terms of order $U(\log c)^{-4}$. The resulting drag correction (involving a two-fold integration over the plane) is given by

$$\frac{F_1^{(3)}}{2\pi\mu} = -\frac{U}{\pi^2 M_{l0}^4}\iint \{K_0^2(x)\,K_0^2(x') + \tfrac{1}{4}K_0^3(x)\,K_0(x')\}\,K_0(|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}\,d\mathbf{x}'$$
$$= -0{\cdot}631\,U/M_{l0}^4,$$

leading to the new self-consistency equation

$$M_{l3} - \tfrac{1}{2}\log M_{l3} + \gamma - \frac{0{\cdot}586}{M_{l3}} + \frac{0{\cdot}714}{M_{l3}^2} + O(M_{l3}^{-3}) = \tfrac{1}{2}\log\frac{2}{c}. \qquad (7.11)$$

Transverse flow

As far as stage 2, the equation corresponding to (7.10) is

$$M_{t2} - \tfrac{1}{2}\log M_{t2} + \gamma - 0{\cdot}470/M_{t2} + O(M_{t2}^{-2}) = \tfrac{1}{2}\log c^{-1}, \qquad (7.12)$$

where the coefficient of M_{t2}^{-1} is given by the leading term in

$$-\frac{\alpha_t^2}{4\pi}\int \mathscr{I}_t^3(\mathbf{r}).\{\mathbf{I} + M_t^{-1}\mathscr{I}_t(\mathbf{r})\}^{-1}\,d\mathbf{r} \qquad (7.13)$$

(for $\mathscr{I}_t$ see the end of § 4).

As with arrays of spheres, the expansions of the W_3 integrals (7.8) and (7.13) are unsatisfactory at practical values of c. So the results were obtained in the way described in § 6, at the end of stage 2, exact expressions being available for W_1 and W_2. This was done for both longitudinal and transverse flows: M_l and M_t are plotted for small values of c in figure 6 and for larger values in figure 7. (The theory would predict infinite drag in longitudinal flow when $c = 1$, and in transverse flow when $c = 0{\cdot}63$; at close packing the value is $c = 0{\cdot}91$.)

It seems that the results for longitudinal flow should be relevant to the study by Batchelor (1971) of the stress generated by pure straining motion in a suspension of elongated particles. Under the conditions described in § 4 of that paper, when the particles are aligned and have lateral spacing small compared with their length, we can differentiate the equation of motion and the boundary conditions with respect to x, the co-ordinate in the direction of the particles. Then the equation and boundary condition for $\partial u/\partial x$ are precisely those for u in the longitudinal flow problem studied here. (When the particles are not very close the averaged equations will be affected by the probability of encountering the end of a particle near the point of observation.) The conclusion would be that a more definite value can be given for the effective ratio of separation to particle radius, h/b in Batchelor's equation (4.8): the denominator $\log(h/b)$ corresponds to M_l in the present paper.

This work was begun during a visit to Cambridge in the first half of 1972. The author wishes to record his gratitude to Professor G. K. Batchelor for this visit to the Department of Applied Mathematics and Theoretical Physics and the use of its facilities, for the introduction to the problem treated here, and discussions about it; also to the secretarial staff of the department. He is indebted, for much valuable exchange of ideas, to Dr E. J. Hinch, Mr D. Jeffrey, and others at Cambridge, and to Dr S. Childress of New York University. The Department of Mathematics at Adelaide University kindly made it possible to use their computing facilities during the final stages of preparation of the paper.

Appendix A. Consequences of spherical symmetry for the form of the velocity, pressure and resistive force fields in incompressible flows with viscosity and resistance linearly dependent on a single vector quantity

The governing differential equations will be (2.1):

$$-\nabla p+\mu\nabla^2\mathbf{u} = -\mathbf{R}, \tag{A 1}$$

with

$$\nabla^2 p = \nabla.\mathbf{R}, \tag{A 2}$$

where the resistive force $\mathbf{R}$ is a linear functional of $\mathbf{u}$. The vector quantity on which the flow depends is denoted by $\mathbf{S}$; its physical meaning is not specified at present.

Discussion is confined to flows generated by forces within a finite region, with a resistance relationship that ensures that $\mathbf{u}$, ∇p and $\mathbf{R}$ are $O(r^{-1})$ for large r.

Now $\mathbf{u}$, p and $\mathbf{R}$ are linear functions of $\mathbf{S}$, and otherwise depend only on $\mathbf{r}$, in such a way that the full rotation–reflexion group applied to $\mathbf{S}$, $\mathbf{r}$, $\mathbf{u}$ and $\mathbf{R}$ leaves the equations satisfied. It follows that $\mathbf{u}(\mathbf{r})$ and $\mathbf{R}(\mathbf{r})$ have the form $\mathbf{r}\mathbf{r}.\mathbf{S}k_1(r)+\mathbf{S}k_2(r)$, and $p(\mathbf{r})$ the form $\mathbf{r}.\mathbf{S}k_3(r)$. These can be put in the more convenient form

$$\mathbf{u} = \mathbf{S}.\nabla\nabla f(r)+\mathbf{S}f_1(r), \quad p = \mu\mathbf{S}.\nabla h(r),$$

$$\mathbf{R} = \mu\mathbf{S}.\nabla\nabla g(r)+\mu\mathbf{S}g_1(r),$$

where the five functions of r can be chosen so that f_1, g_1 and the second derivatives of f, g and h are $O(r^{-1})$ for large r. Then incompressibility requires

$$\mathbf{S}.\nabla\{\nabla^2 f+f_1\} = 0,$$

so that $f_1 = -\nabla^2 f$, given the conditions at infinity. Similarly (A 2) requires $g_1 = \nabla^2(h-g)$. Thus the general form is

$$\mathbf{u} = \mathbf{S}.(\nabla\nabla-\mathbf{I}\nabla^2)f(r), \tag{A 3}$$

$$p = \mu\mathbf{S}.\nabla h(r), \tag{A 4}$$

$$\mathbf{R} = \mu\mathbf{S}.(\nabla\nabla-\mathbf{I}\nabla^2)g(r)+\mu\mathbf{S}\nabla^2 h(r). \tag{A 5}$$

Equation (A 1) leads to

$$\nabla^2 f+g-h = 0, \tag{A 6}$$

since an additional constant can be incorporated into g without any other changes.

From (A 5)

$$\nabla . \mathbf{R} = \mu \mathbf{S} . \nabla \nabla^2 h,$$

so that

$$\nabla . \mathbf{R} = 0 \quad \text{implies} \quad \nabla^2 h = 0. \tag{A 7}$$

This case occurs if $\mathbf{R} = -\mu\alpha^2\mathbf{u}$ (Brinkman), or more generally if

$$\mathbf{R} = -\int \mathbf{u}(\mathbf{r}')\, L(\mathbf{r}-\mathbf{r}')\, d\mathbf{r}'$$

for some function $L(\mathbf{r})$.

Appendix B. Drag produced by a flow satisfying Brinkman's equation, with uniform velocity U at infinity

Pair of equal spheres

The tensor drag coefficient $\mathbf{D}(\mathbf{r}, \alpha)$ defined in § 6 (stage 2) for two equal spheres centred at O and $\mathbf{r}$ is known when $\alpha = 0$. The behaviour when αa is small can conveniently be derived from the integral equation of flow. If $\mathbf{T}(\mathbf{a})$ is the surface stress on one sphere,

$$\int \mathbf{T}(\mathbf{a}) . \mathscr{I}(\mathbf{r}'-\mathbf{a})\, d\mathbf{a} + \int \mathbf{T}(\mathbf{a}) . \mathscr{I}(\mathbf{r}'-\mathbf{r}+\mathbf{a})\, d\mathbf{a} = 8\pi\mu\mathbf{U} \quad \text{on} \quad r' = a \quad (\text{and on } |\mathbf{r}'-\mathbf{r}| = a).$$

Then from the expansion of the Green's function in powers of α [equation (4.8)],

$$\int \mathbf{T}(\mathbf{a}) . \mathscr{I}_0(\mathbf{r}'-\mathbf{a})\, d\mathbf{a} + \int \mathbf{T}(\mathbf{a}) . \mathscr{I}_0(\mathbf{r}'-\mathbf{r}+\mathbf{a})\, d\mathbf{a} = 8\pi\mu\mathbf{U} + \tfrac{8}{3}\alpha \int \mathbf{T}(\mathbf{a})\, d\mathbf{a} + O\left(\alpha^2 r \int |\mathbf{T}(\mathbf{a})|\, d\mathbf{a}\right) \quad \text{on} \quad r' = a. \tag{B 1}$$

Now the integral of $\mathbf{T}$ over the sphere is the total drag force; so if the remainder term is omitted, (B 1) corresponds to unshielded flow past the two spheres, with velocity $\mathbf{U} . \{\mathbf{I} + 2\alpha a \mathbf{D}(\mathbf{r}, \alpha)\}$ at large r', and hence a drag

$$6\pi\mu a \mathbf{U} . \{\mathbf{I} + 2\alpha a \mathbf{D}(\mathbf{r}, \alpha)\} . \mathbf{D}(\mathbf{r}, 0).$$

The remainder term represents a non-uniform flow; the corresponding drag can be estimated on physical grounds as $O(\mu U \alpha^2 a^2 r)$, and thus

$$\mathbf{D}(\mathbf{r}, \alpha) = \{\mathbf{I} + 2\alpha a \mathbf{D}(\mathbf{r}, \alpha)\} . \mathbf{D}(\mathbf{r}, 0) + O(\alpha^2 a r).$$

This can readily be solved for the coefficient $\mathbf{D}(\mathbf{r}, \alpha)$ in terms of $\mathbf{D}(\mathbf{r}, 0)$, giving the expression quoted in § 6.

Pair of equal cylinders parallel to the flow

For the problem of longitudinal flow past two equal parallel infinite cylinders, with axes a distance h apart, a first approximate solution is obtained for the case where h is small compared with the shielding radius. In this approximation the near field is a slow viscous flow without resistance, produced by axial forces $2\pi\mu V$ per unit length on each cylinder. Matching with the far field

$$u = U - 2VK_0(\alpha_l r)$$

enables V to be determined.

The harmonic velocity field having a logarithmic singularity of strength $2V$ at infinity, and vanishing on the two cylinders, can be obtained as a series of reflexions, or in terms of elliptic functions (Morse & Feshbach 1953, p. 1242): this provides the leading approximation to the near field ($\alpha_l r \ll 1$), and is to be matched with the far field in $r \gg h$, giving

$$V \doteqdot U\{K_0(\alpha_l a) + K_0(\alpha_l h) + g(h/a)\}^{-1} \quad \text{if} \quad \alpha_l h \ll 1,$$

where

$$g\left(\frac{h}{a}\right) = \log(1+\eta^{-2}) + 2\log\frac{4\eta^{\frac{3}{2}}\log\eta}{\pi(\eta^2-1)\,\theta_3\theta_4},$$

$$\eta = h/2a + [(h/2a)^2 - 1]^{\frac{1}{2}},$$

$$\theta_3 = \sum_{m=-\infty}^{\infty} \exp\left(-\frac{m^2\pi^2}{2\log\eta}\right), \quad \theta_4 = \sum_{m=-\infty}^{\infty} (-1)^m \exp\left(-\frac{m^2\pi^2}{2\log\eta}\right).$$

In the notation of § 7, stage 2, if $h = |\mathbf{r}|$, V is equal to $UD_l(\mathbf{r}, \alpha_l)$.

It is not yet possible to evaluate the W_4 integral in § 7 from this approximation to $D_l(\mathbf{r}, \alpha_l)$ because the integrand contains a term in r^{-2}, arising from the reflexion of velocity gradients. This is not simply a short-range effect, and must be calculated exactly. It is found then that

$$W_4 = 2cU\int_{2\alpha_l a}^{\infty} \frac{xK_1^2(x)}{\{M_{l0}+K_0(x)\}^2}dx + \frac{2cU}{a^2}$$
$$\times \int_{2a}^{\infty}\left[\frac{1}{M_{l0}+K_0(\alpha_l r)+g(r/a)} - \frac{1}{M_{l0}+K_0(\alpha_l r)} - \frac{a^2 r^{-2}}{\{M_{l0}+K_0(\alpha_l r)\}^2}\right] r\,dr.$$

The last integral now contains only short-range effects, for which the above calculation of D_l is sufficiently accurate.

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